KENYA METHODIST UNIVERSITY
END OF FIRST TRIMESTER 2007 EXAMINATIONS

FACULTY	**:**	**SCIENCES**
DEPARTMENT	**:**	**MATHEMATICS AND COMPUTER SCIENCE**
COURSE CODE	**:**	**MATH 412**
COURSE TITLE	**:**	**ALGEBRAIC STRUCTURES**
TIME	**:**	**2 HRS**

Instructions:
- Answer question 1 and any other 2 questions.

Question 1 (30 marks)

a) Consider the set N of positive integers, and let * denote the operation of least common multiple (LCM) on N.
  i) Find 4*6, 3*5, 9*18 and 1*6. (2 mks)
  ii) Is (N,*) a semigroup? Is it commutative? (2 mks)
  iii) Find the identity of element of* (2 mks)
  iv) Which elements in N, if any, have inverses and what are they? (2 mks)

b) Let H be a finite subgroup of a group G. Show that H and any coset Ha have the same number of elements. (5 mks)

c) In an integral domain D, show that if ab=ac with a≠0 then b=c. (5 mks)

d) Suppose f(t) = $t^3 - 2t^2 - 6t - 3$. Find the roots of f(t) assuming f(t) has an integer root.(5 mks)

e) Prove that in a ring R:
  i) a.0 = 0 (2 mks)
  ii) a(a-b) = (-a)b (2 mks)
  iii) (-1)a = -a (when R has an identity element 1) (3 mks)

Question 2 (20 marks)

a) Consider the group G = $\{1,2,3,4,5,6\}$ under multiplication modulo 7.
  i) Find the multiplication table of G. (3 mks)
  ii) Find 2^{-1}, 3^{-1}, 6^{-1}. (3 mks)
  iii) Find the orders and subgroups generated by 2 and 3. (3 mks)
  iv) Is G cyclic? (1 mk)

b) Prove:
  i) A finite integral domain D is a field. (4 mks)
  ii) Z_p is a field where p is a prime number. (3 mks)
  iii) (Fermat) if p is prime, then $a^p \equiv a$ (mod p) for any integer a. (3 mks)

Question 3 (20 marks)

a) Consider the ring $Z_{10} = \{0,1,2,...,9\}$ of integers module 10.
  i) Find the units of Z_{10} (2 mks)
  ii) Find -3, -8, and 3^{-1} (4 mks)
  iii) Let f(x) = $2x^2 + 4x + 4$. Find the roots of f(x) over Z_{10} (4 mks)

b) Prove the Unique Factorization Theorem which says that if f is a nonzero polynomial in K(t), then f can be written uniquely (except for order) as a product $f=kp_1p_2\ldots p_n$ where $k \in K$ and the p's are monic irreducible polynomials in K(t). (10 mks)

Question 4 (20 marks)

a) Let G be a group and let A be a nonempty set.
  i) Define the meaning of the statement "G acts on A". (3 mks)
  ii) Define the stabilizer Ha of an element a∊A. (3 mks)
  iii) Show that Ha is a subgroup of G. (4 mks)

b) i) Suppose f(t) = $2t^3 - 3t^2 - 6t - 2$. Find all the roots of f(t) knowing that f(t) has a rational root. (6 mks)

ii) Let K = Z_8. Find all roots of f(t) = $t^2 + 6t$ (4 mks)

Question 5 (20 marks)

a) Suppose $f: G \to G'$ is a group homomorphism. Prove:

i) f(e) = e' (5 mks)

ii) $f(a^{-1}) = f(a)^{-1}$ (5 mks)

b) Consider the set Q of rational numbers, and let * be the operation on Q defined by:

$$a*b = a+b - ab$$

i) Find 3*4, 2*(-5) and 7*½ (2 mks)

ii) Is (Q,*) a semigroup? Is it commutative? (3 mks)

iii) Find the identity element for *. (2 mks)

iv) Do any of the elements in Q have an inverse? What is it? (3 mks)