JARAMOGI OGINGA ODINGA UNIVERSITY OF SCIENCE AND TECHNOLOGY

UNIVERSITY EXAMINATIONS
2012/2013 ACADEMIC YEAR
SCHOOL OF MATHEMATICS AND ACTUARIAL SCIENCES

SEMESTER TWO FIRST YEAR EXAMINATIONS

FOR THE DEGREE
OF
MASTER OF SCIENCE (MSc)

COURSE CODE: SMA 842
COURSE TITLE: NUMERICAL ANALYSIS II

DATE : August 2013 **TIME: 3hrs**

INSTRUCTIONS

Attempt any **three** questions

Show all the necessary working

Question1 [20 Marks]

(a) Derive the Trapezoidal method $y_{n+1}=y_n+\frac{h}{2}\left[f\left(x_n,y_n\right)+f\left(x_{n+1},y_{n+1}\right)\right]; h=x_{n+1}-x_n$ step size to approximate the first order initial value problem

$y'=f\left(x,y\right),\ a\le x\le b,\quad y\left(a\right)=\sim$ **[8marks]**

(b) If the first order ordinary differential equation,

$y'=f\left(t,y\right),\ a\le t\le b,\quad y\left(a\right)=\sim$

is approximated by the one-step modified Euler method

$w_0=\sim$

$$w_{i+1}=w_i+\frac{h}{2}\left[f\left(t_i,w_i\right)+f\left(t_{i+1},w_i+hf\left(t_i,w_i\right)\right)\right]:\ \ i=0,1,2,\ldots N-1,$$

prove that modified Euler method is consistent with the ordinary first order differential equation.

[7marks]

(c) Use the Euler's method , with step size $h=0.02$, to estimate $y\left(0.2\right)$ where y satisfies ordinary first order differential equation,

$$y'=\frac{-y^2}{1+x}\ ;\ y\left(0\right)=1$$

On the same table display the results ; n, x_n, y_n **[5marks]**

QUESTION2.[20 marks]

(a) For the second order boundary- value problem

$$y''+\frac{2}{x}y'-\frac{2}{x^2}y=\frac{\sin\left(\ln x\right)}{x^2},\ 1\le x\le 2\quad ;\ \ y\left(1\right)=1,\ y\left(2\right)=2,$$

show that it has a unique solution $y\left(x\right)$ over the sub domain D where

$D=\left\{\left(x,y,y'\right)\middle|\ \ ,\ 1\le x\le 2,-\infty<y<\infty,-\infty<y'<\infty\right\}$ **[8 marks]**

(b) (i) Construct a finite difference scheme to the second order boundary- value problem

$$y''+\frac{2}{x}y'-\frac{2}{x^2}y=\frac{\sin\left(\ln x\right)}{x^2},\ 1\le x\le 2\quad ;\ \ y\left(1\right)=1,\ y\left(2\right)=2,$$

(ii) Deduce that the scheme takes the form

$$-\left(1+\frac{h}{2}p\left(x_i\right)\right)w_{i-1}+\left(2+h^2q\left(x_i\right)\right)w_i-\left(1-\frac{h}{2}p\left(x_i\right)\right)w_{i+1}=-h^2r\left(x_i\right)$$

$: p(x) = -\frac{2}{x}, q(x) = \frac{2}{x^2}, \; r(x) = \frac{\sin(\ln x)}{x^2}$

(iii) Use the above numerical scheme with step-size $h = 0.2$ and an appropriate iterative method to approximate the solution of the above given second order boundary- value problem

On the same table display the results ; n, t_n, y_n, E_n $\quad : E_n = |y(t_n) - y_n|$

Assume the unique solution of the initial value problem is

$$y(t) = 1.13920701320x - \frac{0.03920701320}{x^2} - \frac{3}{10}\sin(\ln x) - \frac{1}{10}\cos(\ln x).$$ **[12 marks]**

QUESTION 3.[20 marks]

Given the mixed boundary- value problem

$y'' = y^3 - yy' \quad ; \quad y'(0) = -1, \; y(1) = 0.5.$

(i) Using step size $h = \Delta x = x_{i+1} - x_i = 0.25$ construct the corresponding finite difference formula to approximate $y(x_n)$ at $x = x_n$. **[5 marks]**

(ii) Solve the boundary value problem applying five times ,a Newton's iterative method. **[15 marks]**

Question.4 [20 marks]

Given the system of linear equations $M\underline{X} = \underline{b}$ where $M = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$,

Prove that Gauss Seidel iterative scheme for the system will converge much faster than Gauss Jacobi's iterative scheme. **[20 marks]**

Question5 [20marks]

(a) Let $y = f(x)$ be a real valued function defined over an interval $[a,b]$

(i) Derive the Simpson's numerical quadrature

$$\int_{x_0}^{x_n} y\,dx \approx \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + ... + y_{n-1}) + 2(y_2 + y_4 + y_6 + ... + y_{n-2})] \; : x_0 = a \;\; x_n = b$$

(ii) Show that the truncation error $R(h)$, in the Simpson's numerical quadrature is given by

$$R(h) = -\frac{h^5}{90} y^{IV}(\P); \qquad \P \in (x_1 - h, x_1 + h)$$ **[8 marks]**

(b) Calculate the approximate value of the improper integral $\int_0^{\frac{1}{2}} \frac{2dx}{\sqrt{x(1-x)}}$ **[12 marks]**

(c) Approximate numerically the integral

$\int_4^{4.4} \int_2^{2.6} \frac{2dxdy}{xy} : on \;\; R = \{(x,y) : 4 \le x \le 4.4, 2 \le y \le 2.6\}$ with : $\Delta x = h = 0.2, \Delta y = k = 0.3$ [6 marks]

Question 1 [20 marks]

Given the system of nonlinear equations

$$f(x,y,z,t)=0$$
$$g(x,y,z,t)=0$$
$$h(x,y,z,t)=0$$
$$k(x,y,z,t)=0$$

(i) Express each functions $f(x,y,z,t), g(x,y,z,t), h(x,y,z,t), k(x,y,z,t)$ about point $p(x_0,y_0,z_0,t_0)$ using Taylor's infinite series as far as the second order terms.

(ii) Derive the iterative formula

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \\ t_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \\ z_n \\ t_n \end{bmatrix} - \begin{bmatrix} f_x & f_y & f_z & f_t \\ g_x & g_y & g_z & g_t \\ h_x & h_y & h_z & h_t \\ k_x & k_y & k_z & k_t \end{bmatrix}^{-1}_{(p_n)} \begin{bmatrix} f \\ g \\ h \\ k \end{bmatrix}_{(p_n)} \quad ; n = 0, 1, 2 \ldots ; p_n = (x_n, y_n, z_n, t_n)$$

(iii) State the Jacobian matrix of the system.

Discus the practical implement of the formula and state any assumptions made. [20 marks]

Question 2 [20 marks]

Suppose the system of non-linear equations

$$6x - 2\cos yz - 1 = 0$$
$$x^2 - 81(y+0.1)^2 + \sin z + 1.06 = 0$$
$$e^{-xy} + 20z + \frac{10® - 3}{3} = 0$$

are defined on $D = \{(x,y,z) : -2 \le x \le 2, -2 \le y \le 2, -2 \le z \le 2\}$ the sub set of R^3.

(i) Determine a mapping $\underline{G} : R^3 \to R^3$ such that $\underline{G}(\underline{x}) = [g_1(\underline{x}), g_2(\underline{x}), g_3(\underline{x})]$, $\underline{x} = (x,y,z)$ is a fixed point for the system .

(ii) show that $\left| \dfrac{\partial g_i(\underline{x})}{\partial x_j} \right| \le \dfrac{K}{3}$; $0 < K < 1$ $\quad i, j = 1,2,3$ and obtain the constant K . $x_1 = x \quad x_2 = y \quad x_3 = z$ .Deduce that the sequence $\left\{ \underline{x}^{(m)} \right\}_{m=0}^{\infty}$; $x_i^{(m)} = g_i(x,y,z)$ converges to the unique fixed point $\underline{p}$ of the system.

iii) Apply at least five times improved fixed point schemes $x_i^{(m)} = g_i(x, y, z)$ to approximate point $\underline{p}$.
Take $(x_0, y_0, z_0) = (0.1, 0.1, -0.10)$. Tabulate results as n

$n,\ x_n\ , y_n\ , z_n\ , ¢_n\ ,\ f_n\ ,\ g_n\ , h_n\ : ¢_n = \left\| \underset{\sim}{X}_{n+1} - \underset{\sim}{X}_n \right\|,\ f_n = f(x_n, y_n, z_n)\ etc$ [20 marks]

Question 3 [20 marks]

For the system of linear equations $A\underline{X} = \underline{b}$ where $A = \begin{bmatrix} 10 & -1 & 3 & 0 \\ -1 & 11 & -1 & 3 \\ 2 & -1 & 10 & -1 \\ 0 & 3 & -1 & 8 \end{bmatrix}$ with $\underset{\sim}{b} = \begin{bmatrix} 6 \\ 25 \\ -11 \\ 15 \end{bmatrix}$

(i) Develop the Jacobi, iterative matrix M_J .
(ii) Write the Jacobi's iterative matrix scheme.
(iii) Compute the spectral radius $°_J$, of M_J and deduce the spectral radius $°_G$ of Seidel's iterative matrix.
Comment appropriately on the nature of convergence of the iterative scheme.

Question 4[20 marks]

Consider the system of non-linear equations

$$f(x, y, z, t) = -2.065x + 2y - 0.0625x^3 + 0.5 = 0$$

$$g(x, y, z, t) = z - 2y + x - 0.0625y^3 + 0.125y(z - x) = 0$$

$$h(x, y, z, t) = t - 2z + y - 0.0625z^3 + 0.125z(t - y) = 0$$

$$r(x, y, z, t) = -1.9375t + z - 0.0625t^3 - 0.125zt + 0.5 = 0$$

(i) Define and obtain the Jacobian matrix J for the system defined at the point (x_n, y_n, z_n, t_n)

(ii) Evaluate $J(x_0, y_0, z_0, t_0)$ show that $J^{-1}(x_0, y_0, z_0, t_0)$ exists where,
$(x_0, y_0, z_0, t_0) = (0.983,\ 0.793, 0.664, 0.571)$

.

(iii) Apply five times the improved Newton's iterative formulae

$$x_{n+1} = x_n - \frac{-2.065x_n + 2y_n - 0.0625x_n^{\ 3} + 0.5}{-2.065 - 0.1875x_n^{\ 2}}$$

$$y_{n+1} = y_n - \frac{z_n - 2y_n + x_{n+1} - 0.0625y_n^{\ 3} + 0.125y_n(z_n - x_{n+1})}{-2 - 0.1875y_n^{\ 2} + 0.125(z_n - x_{n+1})}$$

$$z_{n+1} = z_n - \frac{t_n - 2z_n + y_{n+1} - 0.0625z_n^{\ 3} + 0.125z_n(t_n - y_{n+1})}{-2 - 0.1875z_n^{\ 2} + 0.125(t_n - y_{n+1})}$$

$$t_{n+1} = t_n - \frac{-1.9375t_n + z_{n+1} - 0.0625t_n^{\ 3} - 0.125z_{n+1}t_n + 0.5}{-1.9375 - 0.1875t_n^{\ 2} - 0.125z_{n+1}}$$

to approximate the exact root of the above system.

Take initial root of [1.5, 1.25, 1, 0.75]. Display the results as

$n,\ x_n\ ,y_n\ ,z_n\ ,t_n\ ,¢_n\ ,\ f_n\ ,\ g_n\ ,h_n,r_n\ \ :¢_n = \left\| \underset{\sim}{X}_{n+1} - \underset{\sim}{X}_n \right\|,\ f_n = f\left(x_n, y_n, z_n, t_n\right)\ etc$ [20 marks]

Question5 [20 marks]

Given the 4 by 4 real matrix $M = \begin{bmatrix} 3 & 1 & -2 & -1 \\ 2 & -2 & 2 & 3 \\ 1 & 5 & -4 & -1 \\ 3 & 1 & 2 & 3 \end{bmatrix}$

(i) Determine the Doolittle triangular decomposition of M in the form $M = LU$.

(ii) Obtain the matrix inverse M^{-1}

(iii) Describe the use of the Doolittle triangular decomposition of M in the solution of the linear equation $M\,\underline{X} = \underline{C}\ \ :\underline{X} = \left[x, y, z, w\right]^t$.

CHEBYSHEV POLYNOMIAL FUNCTIONS TABLE

k	$T_k\left(x\right)$
0	1
1	x
2	$2x^2 - 1$
3	$4x^3 - 3x$
4	$8x^4 - 8x^2 + 1$
5	$16x^5 - 20x^3 + 5x$
6	$32x^6 - 48x^4 + 18x^2 - 1$
7	$64x^7 - 112x^5 + 56x^3 - 7x$
n	$T_{n+1}\left(x\right) = 2xT_n\left(x\right) - T_{n-1}\left(x\right)\ \ :T_n\left(x\right) = \cos\left[n\arccos x\right],\ x \in \left[-1,1\right]$
$T_m\left(x\right)T_n\left(x\right) = \frac{1}{2}\left[T_{n+m}\left(x\right) + T_{\lvert n-m\rvert}\left(x\right)\right]$	
$\overline{P}_6\ \left(x\right) = 1.266066T_0 - 1.130318T_1 + .271495T_2 - .044337T_3 + .005474T_4 - .000543T_5 + .000045T_6\left[: e^{-x}\right]$	
$\overline{P}_6\ \left(x\right) = 1.266066T_0 + 1.130318T_1 + .271495T_2 - .044337T_3 + .005474T_4 + .000543T_5 + .000045T_6\left[: e^{+x}\right]$	