MASENO UNIVERSITY

UNIVERSITY EXAMINATIONS 2015/2016

FOURTH YEAR FIRST SEMESTER EXAMINATIONS FOR THE DEGREE OF BACHELOR OF SCIENCE AND BACHELOR OF EDUCATION WITH INFORMATION TECHNOLOGY

MAIN CAMPUS

SMA 401: RING THEORY

Date: 8th January, 2016 Time: 11.00 - 1.00 pm

INSTRUCTIONS:

- Answer question ONE and any other TWO questions.
- Observe further instructions on the answer booklet.

QUESTION ONE. **[Compulsory]** **[30 Marks]**

(a) Give an example of a :

(i) an integral domain that is not a field.

(ii) commutative ring with unity but has zero divisors.

(iii) ring with no unity. **[3 Marks]**

(b)

(i) Define a zero divisor.

(ii) Find all the zero divisors in $\mathbb{Z}_8$. **[4 Marks]**

(c) Prove that the ring $R[x]$ of polynomials in the indeterminate x with coefficients from the ring R is an integral domain if and only if R is an integral domain. **[5 Marks]**

(d) The following subsets of $\mathbb{Z}$ under ordinary addition and multiplication satisfy all but one of the axioms for a ring. In each case state which axiom fails.

(i) The set $S_0 = S \cup \{0\}$ where S is the set of all odd integers .

(ii) The set of nonnegative integers. **[2 Marks]**

(e) (i) Define a field .

(ii) Show that every field is an integral domain. **[7 Marks]**

(f) Factor the polynomial $p(x) = 2x^3 + 4x^2 + 3x + 1$ over $\mathbb{Z}_5$.

[6 Marks]

(g)) Let F be a field and $F[x]$ be the ring of polynomials over F.Show that if $f(x) \in F[x]$ is divided by the factor $x - c$ where $c \in F$ then the remainder is $f(c)$. **[3 Marks]**

QUESTION 2. **[20 Marks]**

(a) Let H be a subset of a Ring R. Prove that H is a subring of R iff H is nonempty and for any $x, y \in H$ the elements $x - y$ and xy are in H.

[8 Marks]

(b) Given that R is a ring with $1 \neq 0$, define a unit in R. Show that a zero divisor can never be a unit. **[5 Marks]**

(c) Given that $\emptyset: R \to R^*$ is a ring homomorphism. Show that:

(i) $\emptyset(-r) = -\emptyset(r)$ for all $r \in$ R.

(ii) If H is a subring of R then $\emptyset(H)$ is a subring of R^*. **[7 Marks]**

QUESTION 3. **[20 Marks]**

(a) Let θ be a ring homomorphism from the ring R to the ring R^*.

(i) Show that the subset of R, defined by $ker\theta = \{x \in R \mid \theta(x) = 0\}$ is a subring of R.

(ii) Prove that ker θ is an ideal of R.

(iii) Show that ker $\theta = \{o\}$ if and only if θ is injective. **[9 Marks]**

(b) Let R be a ring and I be an ideal of R. Define a map $\emptyset$ from R to the quotient ring R / I by $\emptyset(r) = r +$ I. Show that $\emptyset$ is an epimorphism with $ker\emptyset =$ I . **[4 Marks]**

(c) Let $\emptyset$ be as in part (b) above. Define a map from R / I to R^* by $\varphi(a + I) = \emptyset(a)$. Show that φ is an isomorphism from R / I to R^*.

[7Marks]

QUESTION 4. **[20 Marks]**

(a) Let I be a subset of a ring R containing $x - y$, xr and rx for all $x, y \in$ I and $r \in$ R. Prove that I is an ideal of R . **[5 Marks]**

(b) Consider the ring $\mathbb{Z}$ of integers under the usual addition and multiplication. Prove that every ideal in $\mathbb{Z}$ is a principal ideal.

[7 Marks]

(c) Let R be a commutative ring with unity. Show that for any fixed $a \in R$ the set $(a) = \{ ar \mid r \in R \}$ is an ideai of R .

[8 Marks]

QUESTION 5. **[20 Marks]**

(a) Consider the ring $R = \mathbb{Z}/m$ where m is a positive integer.

Show that R is an integral domain if and only if m is prime. **[8 Marks]**

(b) Show that the principal ideal (4) is a maximal ideal of the ring $2\mathbb{Z}$ of even integers. **[5Marks]**

(c) Let R be a field and $p(x)$ be an element in the field F[x].

(i) Define the term irreducible polynomial.

(ii) Prove that $p(x)$ is a unit in F[x] if and only if $p(x)$is a nonzero constant polynomial. **[4 Marks]**

(d) Determine unique factorization of $f(x) = 4x^4 + 2x^3 + 6x^2 + 6x + 3$ in $\mathbb{Z}_7$ expressing it as a product of its prim factors. **[3 Marks]**