MASENO UNIVERSITY

UNIVERSITY EXAMINATIONS 2015/2016

FIRST YEAR FIRST SEMESTER EXAMINATIONS FOR THE DEGREE OF MASTER OF SCIENCE IN PURE MATHEMATICS

MAIN CAMPUS

SMA 808: COMPLEX ANALYSIS II

Date: 14th May, 2016 Time: 9.00 - 12.00 noon

INSTRUCTIONS:

- Answer ANY THREE questions.

QUESTION ONE (20 MARKS)

(a) If $f(z)$ and $g(z)$ ae analytic functions in a domain D and if $f(z) = g(z)$ on a subset S of D which has a limit point in D, show that $f(z) = g(z)$ on the whole of D. [5 marks]

(b) Show that the limit point of the set of poles of a function $f(z)$ is a non-isolated essential singularity. [5 marks]

(c) If $z = a$ is an essential singularity of a function $f(z)$, show that for any arbitrary number l, arbitrary $\epsilon > 0$, and arbitrary $\rho > 0$ there exists a point z in the deleted neighbourhood $0 < |z - a| < \rho$ for which $|f(z) - l| < \epsilon$. [10 marks]

QUESTION TWO (20 MARKS)

(a) Let $\sum f_n(z)$ be a series of functions defined on a closed set S and let $(u_n(z))$ be a sequence of functions defined on S. Let

(i) $u_n(z) \to 0$ uniformly on S

(ii) the sequence $(S_n(z))$ of partial sums of the series $\sum f_n(z)$ be uniformly bounded

(iii) the series $\sum\{u_n(z) - u_{n+1}(z)\}$ is uniformly and absolutely convergent on S.

Show that the series $\sum u_n(z) f_n(z)$ converges uniformly on S. [8 marks]

(b) Test for uniform convergence of the series [8 marks]

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tan \frac{z}{2^n}.$$

(c) Show that the series $\sum \frac{1}{n^z}$ is uniformly convergent in the closed domain $\Re e\, z \geq 1 + \delta$, where $\delta > 0$. [4 marks]

QUESTION THREE (20 MARKS)

(a) Show that the infinite product $\prod(1 + a_n)$ is convergent if the series $\sum \log(1 + a_n)$ is convergent, the principle value of the logarithm being taken in each case. Hence show that $\prod(1 + a_n)$ is convergent if the two series

$$\sum a_n \text{ and } \sum |a_n|^2$$

are both convergent. [10 marks]

(b) Prove that [10 marks]

$$\prod_{n=1}^{\infty} \frac{n^z + n^2 + 1}{n^z + n^2 - 1}$$

represents an analytic function in the domain $\Re e\, z > 2$.

QUESTION FOUR (20 MARKS)

(a) Show that a meromorphic function whose singularity at infinity is at most a pole is necessarily a rational function. [5 marks]

(b) Let

$$z_0 = 0, z_1, z_2, \ldots \qquad (1)$$

be a sequence of complex numbers converging to infinity and let

$$P_n\left(\frac{1}{z - z_n}\right), n = 0, 1, 2, \ldots$$

be a sequence of polynomials in $\frac{1}{z-z_n}$. Show that there exists a meromorphic function $f(z)$ whose poles coincide with the points (1) and whose principle parts at each pole z_n equals $P_n\left(\frac{1}{z-z_n}\right)$. Hence show that if $F(z)$ is a meromorphic function whose poles are given by (1) with residues $P_n\left(\frac{1}{z-z_n}\right)$ at each z_n, then $F(z) = g(z) + f(z)$, where $g(z)$ is an entire function.

[15 marks]

QUESTION FIVE (20 MARKS)

(a) Show that an entire function with no zeros can be expressed in the form $e^{h(z)}$, where $h(z)$ is an integral function [8 marks]

(b) Let $f(z)$ be an entire function with zeros

$$z_0, z_1, z_2, \ldots, z_n, \ldots$$

of orders $m_0, m_1, \ldots, m_n, \ldots$ respectively. Show that there exists an integral function $h(z)$ with no zeros and a sequence of polynomials $(P_n(z))$ such that [12 marks]

$$f(z) = e^{h(z)} \prod \left[\left(1 - \frac{z}{z_n}\right)^{m_n} e^{P_n(z)}\right].$$