MASENO UNIVERSITY

UNIVERSITY EXAMINATIONS 2015/2016

FIRST YEAR FIRST SEMESTER EXAMINATIONS FOR THE DEGREE OF MASTER OF SCIENCE IN PURE MATHEMATICS

MAIN CAMPUS

SMA 804: FUNCTIONAL ANALYSIS II

Date: 7th May, 2016 | Time: 9.00 - 12.00 noon

INSTRUCTIONS:

- Answer ANY THREE questions.

QUESTION ONE (20 MARKS)

(a) Let $(X, \|\cdot\|)$ be a Banach space. Show that a family $\{x_\alpha : \alpha \in \Lambda\}$ of elements of X is summable if and only if for any real $\epsilon > 0$ there exists a finite set $\Pi_\epsilon \subset \Lambda$ such that

$$\left\| \sum_{\alpha \in \Gamma} x_\alpha \right\| < \epsilon$$

whenever Γ is a finite subset of Λ such that $\Gamma \cap \Pi_\epsilon = \varnothing$. [8 marks]

(b) Let $(X, \|\cdot\|)$ be a normed linear space and $T \in B(X)$. Show that the sequence

$$\left(\|T^n\|^{\frac{1}{n}} \right)_{n=1}^{\infty}$$

converges in $(\mathbb{R}, d)$ and that [6 marks]

$$\lim_{n \to \infty} \|T^n\|^{\frac{1}{n}} = \inf \left\{ \|T^n\|^{\frac{1}{n}} : n \in \mathbb{N} \right\}.$$

(c) Let X be a normed linear space, Y a Banach space and S, T be linear transformations from X to Y such that $D_S \supseteq D_T$ (D_S is the domain of S, etc.). If S is bounded and T is closed show that $S + T$ is closed. [6 marks]

QUESTION TWO (20 MARKS)

(a) Let X be a Banach space and Y a n.l.s. Let $\{T_\alpha : \alpha \in \Lambda\}$ be a nonvoid family of $B(X, Y)$ such that

$$\sup \{\|T_\alpha x\| : \alpha \in \Lambda\} < \infty$$

for all $x \in X$. Show that [10 marks]

$$\sup\{\|T_\alpha\| : \alpha \in \Lambda\} < \infty.$$

(b) In a n.l.s. X show that every weak Cauchy sequence is bounded. [5 marks]

(c) Let (T_n) be a sequence of bounded linear transformations defined on a Banach space X to a n.l.s. Y and suppose that $s\text{-}\lim T_n x$ exists at each $x \in X$. If we define a map T on X by

$$Tx = s\text{-}\lim T_n x$$

such that $T \in B(X, Y)$ and that $\|T\| \leq \underline{\lim}_{n \to \infty} \|T_n\|$. [5 marks]

QUESTION THREE (20 MARKS)

(a) State the open mapping theorem and use it to prove

- (i) Banach's inverse theorem [3 marks]
- (ii) Closed graph theorem [5 marks]

(b) Let X be a linear space over $\mathbb{K}$ and $\|\cdot\|_1, \|\cdot\|_2$ are two Banach space norms for X. If there is a positive constant K such that

$$K\|x\|_1 = \|x\|2 \text{ for all } x \in X,$$

show that the two norms are equivalent. [6 marks]

(c) Let X be a Banach space and Y a n.l.s. and $T : D_Y \to Y$ be a closed linear transformation, where D_T is a linear subspace of X which is not $\{\bar{0}\}$. If T is bounded from below, show that the range R_T of T is closed. [6 marks]

QUESTION FOUR (20 MARKS)

(a) Let M be a closed linear subspace of a Hilbert space H and let $x \in M$. Let $d = \inf\{\|y - x\| : x \in M\}$. Show that there exists a unique element $y_0 \in M$ such that $\|y_0 - x\| = d$ and that $x - y_0 \perp M$. [10 marks]

(b) State and prove the Riesz representation theorem for a Hilbert space. [10 marks]

QUESTION FIVE (20 MARKS)

(a) Let H, K be Hilbert spaces and $T \in B(K, H)$. Show that there exists a unique $T^* \in B(K, H)$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \text{ for all } (x, y) \in H \times K.$$

Also show that $\|T^*\| = \|T\|$ and $(T^*)^* = T$ [10 marks]

(b) H is an inner product space (i.p.s.) and $E = \{z_\alpha : \alpha \in \Lambda\}$ is an orthonormal set of vectors in H. Show that [5 marks]

$$\sum |\langle y, z_\alpha\rangle|^2 \leq \|y\|^2 \text{ for all } y \in H.$$

(c) Let H be a Hilbert space with an infinite orthonormal set F. Show that E can never be a Hamel base for H. [5 marks]