MASENO UNIVERSITY

UNIVERSITY EXAMINATIONS 2016/2017

THIRD YEAR FIRST SEMESTER EXAMINATION FOR DEGREE OF BACHELOR OF SCIENCE IN PHYSICS AND MATERIALS SCIENCE WITH INFORMATION TECHNOLOGY

MAIN CAMPUS

SPH 303: QUANTUM MECHANICS I

Date: 9th December, 2016 Time: 8.30 - 11.30am

INSTRUCTIONS:

- Answer Question ONE and any other TWO.

Question One

(a) (i) Calculate the de Broglie wavelength of a free particle of mass $2gm$ and kinetic energy $100J$. Express the answer in terms of the Planck constant h

(4 marks)

(ii) The wave function of a two-state particle is obtained as a superposition $\Psi = C_1(t)\psi_1(\vec{r}) + C_2(t)\psi_2(\vec{r})$ where $\psi_1(\vec{r})$ and $\psi_2(\vec{r})$ are orthonormal state functions with respective time-dependent probability amplitudes $C_1(t)$ and $C_2(t)$. Show that the state probability amplitudes satisfy normalization condition

$|C_1(t)|^2 + |C_2(t)|^2 = 1$

(6 marks)

(b) (i) Write down the Hermitian operators of position vector $\vec{r}$, linear momentum $\vec{p}$ and total energy E of a particle **(3 marks)**

(ii) Given the total energy conservation equation

$$E = \frac{p^2}{2m} + V(\vec{r})$$

for a particle of mass m, linear momentum $\vec{p}$, potential energy $V(\vec{r})$ and total energy E located at position $\vec{r}$ from the origin, apply quantum operator definitions to obtain the appropriate quantum mechanical equation for the particle

(5 marks)

(iii) Derive the quantum commutation relation between the position coordinate operator $\hat{x}$ and kinetic energy operator $\frac{\hat{p}^2}{2m}$ for a particle moving along the x-axis, where $\hat{p}$ is the linear momentum **(4 marks)**

(c) The wave function of a free particle of mass $3gm$ moving along the x-axis is obtained in the form

$$\psi(x) = \frac{1}{\sqrt{2}}(e^{-ikx} + e^{ikx})$$

where k is the wave number and x is displacement. Determine the kinetic energy of the particle, if it is confined in the region $x = 0 \rightarrow x = 2cm$. **(8 marks)**

Question Two

The Hamiltonian of linear harmonic oscillator of mass m and angular frequency ω with displacement x and linear momentum p along the x-axis is expressed operator form $H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$

(a) (i) Given the Schroedinger equation $H\psi = E\psi$ where E is the total energy. Apply a factorization procedure to show that

$$\frac{\int_{-\infty}^{\infty} \hbar\omega|\hat{a}\psi|^2 \, dx}{\int_{-\infty}^{\infty}|\psi|^2 \, dx} = E - \frac{1}{2}\hbar\omega$$

where the operator $\hat{a}$ and its Hermitian conjugate $\hat{a}^\dagger$ are to be determined through the factorization of the Hamiltonian. **(9 marks)**

(ii) Determine the ground state wave function of the oscillator **(4 marks)**

(b) Calculate the uncertainty in the measurement of the quadrature component

$$\hat{a}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger)$$

in the ground state. Useful integrals:

$$\int_{-\infty}^{\infty} xe^{-\alpha x^2} dx = 0 \quad ; \quad \int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}} \quad ; \quad \int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$$

(7 marks)

Question Three

Consider a particle of mass m moving in a one-dimensional field potential $V(x)$ governed by the Schroedinger equation

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right)\psi = E\psi$$

(a) (i) Express the Schroedinger equation as an equation for a particle in a scattering potential **(3 marks)**

(ii) By using a factorization procedure, determine the scattering amplitude **(8 marks)**

(b) If the particle exits the scattering region as a plane wave with final wave number k_f, determine the scattering cross-section in a scattering potential of the form

$V(x) = e^{-ax}\cos bx$

Useful integral

$$\int_{-\infty}^{\infty} e^{-ax}\cos\beta x dx = \frac{a}{a^2+b^2}$$

(9 marks)

Question Four

(a) For a time-independent Hamiltonian H, the time-dependent Schroedinger equation has solution

$$\Psi(t,\vec{r}) = e^{-\frac{i}{\hbar}Ht}\psi(\vec{r})$$

where the position-dependent wave function $\psi(\vec{r})$ satisfies an eigenvalue equation

$$H\psi(\vec{r}) = E\psi$$

Determine the physical meaning of the constant quantity E **(5 marks)** (b)(i) Obtain the general solution of the free particle Schroedinger equation

$$-\frac{d^2\psi}{dx^2} = E\psi$$

(9 marks)

(ii) Normalize the wave function if the particle is confined within the region $x=-a \to x=a$, along the x-axis (iii) Show that the kinetic energy of the particle is quantized **(6 marks)**

Question Five

(a) Show that in a slowly varying potential $V(x)$, the general solution of the Schroedinger equation in the classically allowed region where the total energy E is greater than the potential energy may be obtained in an approximate form

$$E > V(x) \quad : \quad \psi(x) = \frac{A}{\left(2m(E-V(x))\right)^{\frac{1}{4}}}\cos\left(\int_0^x \sqrt{2m(E-V(y))}\,dy + \theta\right)$$

where θ is a constant phase angle **(14 marks)**

(b) Explain the expected behavior of the wave function as the particle approaches the boundary between the classically allowed region and the classically forbidden region where the total energy is less than the potential energy, $E < V(x)$. **(6 marks)**