UNIVERSITY OF KABIANGA
SCHOOL OF SCIENCE AND TECHNOLOGY
THIRD YEAR FIRST SEMESTER MAIN
EXAMINATION FOR THE DEGREE OF BACHELOR
OF SCIENCE, BACHELOR OF SCIENCE (APPLIED
STATISTICS WITH COMPUTING) AND BACHELOR
OF EDUCATION

MAT 310: REAL ANALYSIS

INSTRUCTIONS:

1. This paper consists of FIVE questions

2. Attempt Question 1 and any other TWO Questions

3. Observe further instructions on the answer booklet.

QUESTION 1 (Compulsory) [30 Marks]

(a) Define the following concepts as used in Real Analysis.

i) Convergent sequence [2 marks]

ii) Adherent point [2 marks]

iii) Enumerable set [2 marks]

(b) Let $\mathbb{R}$ be the set of real numbers. For each $x_1, x_2 \in \mathbb{R}$, show that the function $d : \mathbb{R} \times \mathbb{R}$ defined by $d(x_1, x_2) = \mid x_1^2 - x_2^2 \mid$ is pseudometric on $\mathbb{R}$. [3 marks]

(c) Find the supremum and the infimum of the set

$$\{1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{2^2}, ..., 1 + \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^{n-1}}\}$$

[2 marks]

(d) i) What is a derived set? [1 mark]

Hence or otherwise, find the derived set of each of the following

ii) A finite set [1 mark]

iii) The set $\{(-1)^n + \frac{1}{n} : n \in \mathbb{N}\}$ [1 mark]

iv) The set of irrational numbers. [1 mark]

(e) Show that the sequence (y_m) defined by

$$(y_m) = (\sqrt{m+2} - \sqrt{m+1}), \forall m \in \mathbb{N}$$

is convergent. [4 marks]

(f) Let (x_n) converge to m_1 and (y_n) converge to m_2. Show that

$$(x_n + y_n) \to m_1 + m_2.$$

[3 marks]

(g) Evaluate $lim_{x \to \infty}(\frac{x^3+1}{x^3-3})^{\frac{x^3}{5}}$. [4 marks]

(h) i) Is every bounded sequence convergent? [1 mark]

ii) Support your answer in (i) above. [3 marks]

QUESTION 2 [20 Marks]

(a) Let P^* be a refinement of a partition P. Show that, for a bounded function f, the upper Darboux sums $U(P^*, f) \leq U(P, f)$. [6 marks]

(b) State and prove the D'Alembert's Ratio Test for convergence of a series. [10 marks]

(c) Show that the union of a finite number of closed sets is closed. [4 marks]

QUESTION 3 [20 Marks]

(a) Show that every subset of a countable set is countable. [8 marks]

(b) Show that the function $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[a, \infty)$ where $a > 0$, but not uniformly continuous on $(0, \infty)$. [12 marks]

QUESTION 4 [20 Marks]

(a) Show that for each natural number n, a sequence (x_n) has a limit x if its limit superior coincides with the limit inferior. [5 marks]

(b) Show that the constant function $f(x) = k$ is Riemann integrable and that $\int_a^b f(x)dx = (b-a)k$ for $b \geq a$. [5 marks]

(c) Show that the necessary and sufficient conditions for a real number t to be the supremum of a bounded set S are the following:
i) $x \leq t, \forall x \in S$,
ii) for each positive real number ϵ, there exists a real number $x \in S$, such that $x > t - \epsilon$. [10 marks]

QUESTION 5 [20 Marks]

(a) Show that every open interval is an open set. [5 marks]

(b) Test the convergence of the series $(\frac{2^2}{1^2} - \frac{2}{1})^{-1} + (\frac{3^3}{2^3} - \frac{3}{2})^{-2} + ...$ [5 marks]

(c) i) State and prove the necessary condition for the convergence of a series. [6 marks]

ii) Prove that the converse of the condition in (i) is not true. [4 marks]