SOUTH EASTERN KENYA UNIVERSITY

UNIVERSITY EXAMINATION 2015/2016

FOURTH YEAR FIRST SEMESTER EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE (STATISTICS)

STA 401: MEASURE, PROBABILITY AND INTEGRATION

DATE: 16TH APRIL 2015 TIME: 2 HOURS

Instructions: Answer question one and any other two questions.

QUESTION ONE. (30 MARKS)

a) i) Define a measure space. (2 marks)

ii) When is a measure space called complete? (3 marks)

b) State three necessary and sufficient conditions for a function f to be measurable. (3 marks)

c) If $\{g_n\}$ is a sequence of measurable functions, prove that the functions $\lim_n sup\ g_n$ and $\lim_n \inf\ g_n$ are measurable. (3 marks)

d) Let $f : \mathcal{X} \rightarrow \mathbb{R}$ be measurable and suppose that $f \geq 0$. When is the function f not integrable? (4 marks)

e) Prove that the distribution function $\mathcal{F}_f$ has the following properties:

i) $0 \leq \mathcal{F}_f(x) \leq 1$ for all $x \in \mathbb{R}$ (1 mark)

ii) $\mathcal{F}_f(x) \leq \mathcal{F}_f(y)$ where $x \leq y$ (3 marks)

iii) $\lim_{n\to\infty} \mathcal{F}_f(x) = 0$ and $\lim_{n\to\infty} \mathcal{F}_f(x) = 1$ (5 marks)

iv) $\mathcal{F}_f$ is continuous from the right, that is, for each $x \in \mathbb{R}$, we have $\mathcal{F}_f(x) = \lim_{h\downarrow 0} \mathcal{F}_f(x+h)$ (5 marks)

f) What is a bounded sequence? (1 mark)

QUESTION TWO (20 MARKS)

a) If μ is a finite measure on Σ, prove that the following hold.

i) $\mu(\phi) = 0$ (3 marks)

ii) If $A_1, \dots, A_n \in \Sigma$ with $A_i \cap A_j = \phi$ for $i \neq j$, then $\mu(A_1 + A_2 + \dots + A_n) = \mu(A_1) + \mu(A_2) + \dots + \mu(A_n)$ (2 marks)

iii) If $A, B \in \Sigma$ with $A \subseteq B$, then $\mu(A) \leq \mu(B)$ (2 marks)

iv) If $A1 \subseteq A2 \subseteq \dots$ with $An \in \Sigma$, for n = 1, 2, …, then we have $\mu(A_n) \uparrow \mu(\bigcup_m A_m)$ as $n \to \infty$ (5 marks)

v) If $A_1 \supseteq A_2 \supseteq \dots$ with $An \in \Sigma$ for n = 1, 2, …, then we have $\mu(A_n) \downarrow (\bigcap_m A_m)$ as $n \to \infty$ (4 marks)

vi) Prove that Σ^1 is a σ – algebra. (4 marks)

QUESTION THREE (20 MARKS)

a) If g_1 and g_2 are measurable functions on a common domain, then prove that, each of the following sets is measurable. (10 marks)

i) $\{\omega : g_1(\omega) < g_2(\omega)\}$

ii) $\{\omega : g_1(\omega) = g_2(\omega)\}$

iii) $\{\omega : g_1(\omega) > g_2(\omega)\}$

b) If $\{g_n\}$ is a sequence of measurable functions, prove that $\sup_n g_n$ and $\inf_n g_n$ are measurable. (10 marks)

QUESTION FOUR (20 MARKS)

Suppose that f, g are measurable and let E ϵ Σ. Then prove the following:

i) If $0 \leq f \leq g$, then $\int_E f\, d\mu \leq \int_E g\, d\mu$ (3 marks)

ii) If A ⊆ B, for A, B ϵ Σ and $f \geq 0$ then $\int_A f\, d\mu \leq \int_B f\, d\mu$ (2 marks)

iii) If $f \geq 0$ and c $\geq$ 0 is a constant, then $\int_E cf\, d\mu = c\int_E f\, d\mu$. (3 marks)

iv) If $f = 0$ for all $x \,\epsilon$ E, then $\int_E f\, d\mu = 0$ (3 marks)

v) If μ (E) = 0, then $\int_E f\, d\mu = 0$ for any $f \geq 0$. (2 marks)

vi) If $f \geq 0$, then $\int_E f\, d\mu = \int_X 1_E\, d\mu$ (7 marks)

QUESTION FIVE (20 MARKS)

a) For any $a \,\epsilon\, \mathbb{R}$, prove that, the jump of $\mathcal{F}_f$ at a is equal to $\mathbb{P}$ ($f = a$). (7 marks)

b) Prove that, the non-zero jumps of the distribution function $\mathcal{F}_f$ form a countable set. (5 marks)

c) Prove that, for any random variable f, there is a countable set $J \,\epsilon\, \mathbb{R}$ such that $\mathbb{P}$ ($f = x$) = 0 for all $x \,\epsilon\, \mathbb{R}/J$. (2 marks)

d) Construct a non-decreasing function $\mathcal{F}(x)$ on $\mathbb{R}$, obeying $0 \leq \mathcal{F}(x) \leq 1$, by means of a certain limiting procedure.

Take $\mathcal{F}(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 \leq x \leq 1 \\ 1, & x \geq 1 \end{cases}$

Construct $\mathcal{F}_1(x), \mathcal{F}_2(x), \mathcal{F}_3(x)$.

Explain the shapes. (6 marks)