MASENO UNIVERSITY

UNIVERSITY EXAMINATIONS 2016/2017

SECOND YEAR FIRST SEMESTER EXAMINATIONS FOR THE DEGREE OF BACHELOR OF SCIENCE WITH INFORMATION TECHNOLOGY

MAIN CAMPUS

MMA 221: INTRODUCTION TO NUMBER THEORY

Date: 8th December, 2016 Time: 8.30 - 11.30 am

INSTRUCTIONS:

- Answer question ONE and any other TWO questions.

QUESTION ONE (Compulsory) [30 Marks]

(a) Differentiate between a multiplicative function and a completely multiplicative

QUESTION ONE (Compulsory) [30 Marks]

(a) Differentiate between a multiplicative function and a completely multiplicative function. [2 Marks]

(b) Let $a, b, c,$ and d be integers such that $a \equiv b \mod m$ and $c \equiv d \mod m$. Show that $ac \equiv bd \mod m$. [3 Marks]

(c) Use Chinese Remainder Theorem to solve the following system of linear congruences:

$$\begin{aligned} 7x + 2 &\equiv 1 \mod 3 \\ 3x &\equiv 4 \mod 10 \\ 8x &\equiv 5 \mod 7 \end{aligned}$$

[5 Marks]

(d) Calculate the following:

(i) Euler's totient function, $\phi(72)$. [2 Marks]

(ii) Number of divisors, $d(180)$. [2 Marks]

(iii) Möbius function, $\mu(30)$. [2 Marks]

(e) State Fermat's Little Theorem and hence use it to find $7^{222} \mod 11$. [4 Marks]

(f) Find the solutions of the quadratic congruence $3x^2 + x + 1 \equiv 0 \mod 5$. [3 Marks]

(g) Let S_2 be the set consisting of the sums of two squares. Show that S_2 is closed under multiplication. [3 Marks]

(h) Obtain a continued fraction representation of $\frac{2016}{1963}$. [4 Marks]

QUESTION TWO [20 Marks]

(a) State the law of quadratic reciprocity. [2 Marks]

(b) Use quadratic reciprocity to show that the quadratic congruence $x^2 \equiv 271 \mod 23$ has a solution and hence solve it. [6 Marks]

(c) Let a, b be positive integers satisfying $a^j = b^k$ for positive and relatively prime integers j, k. Show that $a = r^k$ and $b = r^j$ for some positive integer r. [4 Marks]

(d) Prove that if $f(n)$ is a multiplicative function, then

$$g(n) = \sum_{k|n} f(k) \quad \text{and} \quad h(n) = \sum_{k^2|n} f(k^2)$$

are also multiplicative. [8 Marks]

QUESTION THREE [20 Marks]

(a) If $p \equiv 3 \mod 4$ then p cannot be written as a sum of two squares. [3 Marks]

(b) Show that if two integers a and b can be represented as a sum of two squares, then ab can be written as a sum of two squares as well. [4 Marks]

(c) Show that if a positive integer a is represented in decimal digits, i.e.,

$$a = a_n 10^n + a_{n-1} 10^{n-1} + \cdots + a_2 10^2 + a_1 10 + a_0,$$

then a is divisible by

(i) 2 if a_0 is divisible by 2. [2 Marks]

(ii) 3 if the sum of its digits is divisible by 3. [2 Marks]

(iii) 11 if the alternating sum of its digits is divisible by 11. [2 Marks]

(d) Let a, b, x and y be integers. Prove that

(i) if $x|ay$, $x|by$ and $\gcd(a, b) = 1$ then $x|y$. [3 Marks]

(ii) $\gcd(a, b) \cdot \text{lcm}(a, b) = ab$. [4 Marks]

QUESTION FOUR [20 Marks]

(a) Show that there are no positive integers x, y and z such that $x^4 + y^4 = z^2$. [8 Marks]

(b) Use continued fractions to:

(i) solve the Diophantine equation $214x - 35y = 1$. [5 Marks]

(ii) find the greatest common divisor of 214 and 35. [3 Marks]

(c) Prove that there are infinitely many primes. [4 Marks]

QUESTION FIVE [20 Marks]

(a) Show that if $a_1, a_2, \ldots, a_m$ is a complete residue system modulo m and $\gcd(k, m) = 1$, then $ka_1, ka_2, \ldots, ka_m$ is also such a system. [4 Marks]

(b) (i) Define a Pell equation. [1 Mark]

(ii) Solve the following Diophantine equation: $2X^2 + Y^2 = 688$. [5 Marks]

(c) Compute the following:

(i) $\left(\frac{83}{103}\right)$, where $\left(\frac{a}{b}\right)$ is a Legendre symbol. Give a reason for each step. [7 Marks]

(ii) $\sigma_2(144)$, where $\sigma_k(n)$ is the sum of k-th powers of the divisors of n. [3 Marks]