MASENO UNIVERSITY

UNIVERSITY EXAMINATIONS 2016/2017

THIRD YEAR FIRST SEMESTER EXAMINATION FOR DEGREE OF BACHELOR OF SCIENCE, BACHELOR OF ARTS AND BACHELOR OF EDUCATION WITH INFORMATION TECHNOLOGY

CITY CAMPUS - EVENING

MMA 300: REAL ANALYSIS I

Date: 22nd November, 2016 Time: 9.00 - 12.00pm

INSTRUCTIONS:

- Answer Question ONE and any other TWO
- Electronic scientific calculators may be used
- Proofs should be written carefully
- Observe further instructions on the answer booklet
- Cheating is NOT allowed and will be harshly punished

QUESTION ONE (Compulsory) [30 Marks]

(a) Define the following terms as used in analysis: [2 mks]

(i) Monotonic sequence

(ii) Convergent sequence

(b) Let $f(x)$ be a real-valued function defined on an interval $I = \mathbb{R}$.

(i) What is meant by "f is uniformly continuous on I"? [2 mks]

(ii) If $f(x) = x + \sin x$, show that f is uniformly continuous on I. [3 mks]

(c) State the cardinalities of the following sets: [2 mks]

(i) the empty set $\emptyset$

(ii) a singleton set

(iii) the set of all natural numbers $\mathbb{N}$

(iv) the set of all real numbers $\mathbb{R}$

(d) Define a Cauchy sequence and determine whether the sequence (x_n) where $x_n = 1 + \frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \cdots + \frac{1}{7^n}$ is a Cauchy sequence. [4 mks]

(e) If $\sum_n x_n$ is a series of positive terms, show that the sequence of its partial sums (S_n) is monotonically increasing. [3 mks]

(f) Find the lim inf and lim sup of the sequence (x_n) where [3 mks]

$$x_n = \begin{cases} 3^{(1-\frac{1}{n+1})} & \text{if } \; n \text{ is odd,} \\ (-1)^{n+1} & \text{if } \; n \text{ is even.} \end{cases}$$

(g) Using the transitive property of a metric d, show that on any nonempty set X, a metric d is always nonnegative. [2 mks]

(h) Let a function f be given by

$$f(x) = \begin{cases} 1 & \text{when } \; x \text{ is rational,} \\ 0 & \text{when } \; x \text{ is irrational.} \end{cases}$$

Show that f is not Riemann integrable an any interval $[a, b]$. [4 mks]

(i) Determine the convergence/ divergence of the series [3 mks]

$$\frac{1}{3^2} \cdot \frac{2}{4^2} + \frac{3}{5^2} \cdot \frac{4}{6^2} + \frac{5}{7^2} \cdot \frac{6}{8^2} + \cdots$$

(j) Let $|A|$ and $|B|$ denote the cardinalities of the sets A and B respectively. Give an equivalent statement to the statement $|A| = |B|$. [2 mks]

QUESTION TWO [20 Marks]

(a) Distinguish between the following terms and give examples of each: [8 mks]

(i) Absolute and Conditional convergences

(ii) Countable and Uncountable sets

(b) Show that the function $f(x) = \frac{1}{x}$ is continuous but not uniformly continuous on the interval $(0, 1]$. [7 mks]

(c) Use D'Alembert's Ratio test to determine the convergence or divergence of the series

$$\sum_{n=1}^{\infty} n^2 \cdot \sin(\frac{\pi}{2^n})$$

[5 mks]

QUESTION THREE [20 Marks]

(a) Prove that every convergent sequence (x_n) is bounded. With the help of an example, show that the converse is not necessarily true. [8 mks]

(b) Investigate the convergence or divergence of the series below: [8 mks]

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

(c) Let (X, d) be a metric space. Show that every open sphere (ball) in (X, d) is an open set. [4 mks]

QUESTION FOUR [20 Marks]

(a) Show that the function $f(x) = 3x^2 + 2$ is Riemann integrable on the interval $[0, 2]$, and that $\int_0^2 f(x)\, dx = 12$. [8 mks]

(b) Let P^* be a refinement of a partition P of an interval $[a, b]$. Show that for a bounded function f on $[a, b]$,

$$U(P, f) \geq U(P^*, f),$$

where $U(P, f)$ and $U(P^*, f)$ are the upper Riemann sums with respect to the partitions P and P^* respectively. [7 mks]

(c) Show that the intersection of any two open sets in a metric space is also an open set. [5 mks]

QUESTION FIVE [20 Marks]

(a) Show that the set $\mathbb{R}$ of all real numbers is uncountable. [7 mks]

(b) Using the definition of a limit, show that [3 mks]

$$\lim_{n\to\infty} \frac{2n}{3n-1} = \frac{2}{3}.$$

(c) Evaluate the following limit: [3 mks]

$$\lim_{n\to+\infty} \left(\frac{2n+4}{2n-2}\right)^{5n+1}$$

(d) Let $C(a,b)$ be the space of continuous real-valued functions on the interval (a,b). Define $d : C(a,b) \times C(a,b) \to \mathbb{R}$ by

$$d(f,g) = \sup\{|f(x) - g(x)| : x \in (a,b)\}.$$

Show that d defines a metric on $C(a,b)$. [7 mks]

END
ALL THE BEST