MASENO UNIVERSITY

UNIVERSITY EXAMINATIONS 2016/2017

FOURTH YEAR FIRST SEMESTER EXAMINATIONS FOR THE DEGREE OF BACHELOR OF SCIENCE AND BACHELOR OF EDUCATION WITH INFORMATION TECHNOLOGY

MAIN CAMPUS

MMA 401: RING THEORY

Date: 30th November, 2016 Time: 8.30 - 11.30 am

INSTRUCTIONS:

- Answer question ONE and any other TWO questions.
- Observe further instructions on the answer booklet.

Question 1. **(Compulsory)** **[30 Marks]**

(a) Define and give an example of;

(i) An integral domain.

(ii) A commutative ring.

(iii) A zero divisor. **[6Marks]**

(b)(i) Prove the uniqueness of the identity element in a ring R.

(ii) Let x , y and z be elements of an integral domain with $x \neq 0$ and $xy = xz$. Show that $y = z$.

[5Marks]

(c) Give an example of ;

(i) A commutative ring with unity but has zero divisors.

(ii) A non commutative ring with unity.

(iii)A finite field. **[6Marks]**

(d) (i) Define an ideal of a ring R.

(ii) Let R be a commutative ring with unity. Show that for any fixed a in R , the set $(a) = \{ar | r \in R\}$ is an ideal of R. **[8Marks]**

(e)(i) Define the term irreducible polynomial over a field F.

(ii) Determine whether or not the polynomial $g(x) = x^2 + x + 2$ is irreducible over $\mathbb{Z}_5$ and over $\mathbb{Z}_7$. **[5Marks]**

Question 2.

(a) Define

(i) principal ideal.

(ii) maximal ideal. **[4 Marks]**

(b) (i)Show that the set $R = \left\{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| a, b, c \in \mathbb{Z}\right\}$is a ring with respect to matrix addition and multiplication.

(ii) Prove that the sets $I = \left\{\begin{bmatrix} p & q \\ 0 & 0 \end{bmatrix} \middle| p, q \in \mathbb{Z}\right\}$ is an ideal of R.

[16 Marks]

Question 3.

(a) For any $a \in \mathbb{Z}$, let $[a]_6$ denote $[a]$ in $\mathbb{Z}_6$ and $[a]_2$ denote $[a]$ in $\mathbb{Z}_2$.

(i)Prove that the mapping $\Phi: \mathbb{Z}_6 \longrightarrow \mathbb{Z}_2$ defined by $\Phi([a]_6) = [a]_2$ is a homomorphism.

(ii) Find the kernel of Φ. **[4 Marks]**

(b) Given that $\phi: R \longrightarrow \tilde{R}$ is a homomorphism. Show that,

(i) $ker\phi = \{x \in R: \phi(x) = 0\}$ is an ideal of R.

(ii) $ker\phi = \{0\}$ iff ϕ is injective. **[7 Marks]**

(c) Consider the ring $\mathbb{Z}$ of integers. Prove that every ideal of $\mathbb{Z}$ is of the form

$(a) = \{ar \mid r \in \mathbb{Z} \text{ and } a \text{ is fixed in } \mathbb{Z}\}$ **[9Marks]**

Question 4

(a)Give the definition of;

(i) a subring

(ii) ring homomorphism. **[4 Marks]**

(b) Let φ be a homomorphism from the ring R to a ring $\bar{R}$. Let R_1 and $\bar{R}_1$ be subrings of R and R_1 respectively. Prove that

(i) $\varphi(R_1)$ is a subring of $\bar{R}$.

(ii) $\varphi^{-1}(\bar{R}_1)$ is a subring of R. **[~~12~~ 16 Marks]**

Question 5.

(a) Definition of a field and hence prove that every field is an integral domain.

[8 Marks]

(b) Let R be an integral domain. If $f(x)$ and $g(x)$ are non zero elements of $R[x]$, the set of all polynomials over R. Show that

$\deg(f(x).g(x)) = \deg(f(x)) + \deg(g(x))$. **[5 marks]**

(c) A polynomial $f(x)$ is known to be of degree one or two over F. Prove that $f(x)$ is reducible over F if and only if it has at least one zero in F.

[7 Marks]