MASENO UNIVERSITY

UNIVERSITY EXAMINATIONS 2016/2017

FOURTH YEAR FIRST SEMESTER EXAMINATIONS FOR THE DEGREE OF BACHELOR OF SCIENCE IN APPLIED STATISTICS, ACTUARIAL SCIENCE, MATHEMATICAL SCIENCE AND MATHEMATICS AND ECONOMICS WITH INFORMATION TECHNOLOGY

MAIN CAMPUS

MMA 417: GROUP THEORY II

Date: 28th November, 2016 Time: 12.00 - 3.00 pm

INSTRUCTIONS:

- Answer question ONE and any other TWO questions.
- Observe further instructions on the answer booklet.

QUESTION 1. (COMPULSORY) (30 MARKS)

(a) Let A_3 be the subgroup $\{I, (123), (132)\}$ of S_3 the subgroup of S_4 of permutations that fix 4. Let D_4 be the group of symmetries of a square. Show that $S_4 = A_3 D_4$. [3 marks]

(b) Let G be a group. If p is a prime integer dividing $|G|$, then G contains an element of order p. Use this fact to show that if every element of G has order a power of p, then G is a p-group. [2 marks]

(c) Let H and K be subgroups of G. Show that the number of distinct H-conjugates of K is $|H : N_G(K) \cap H|$. [5 marks]

(d) Prove that if a non-abelian group G has order p^3 where p is prime, then G has nilpotence class 2. [5 marks]

(e) Show that every finite abelian group G is the direct product of its Sylow subgroups. [6 marks]

(f) Given that the derived subgroup G' of a group G is normal in G. Show that G' is the unique minimal normal subgroup N with G/N abelian, precisely G/N is abelian if and only if $G' \leq N$. [3 marks]

(g) State the second Sylow's theorem and use it to prove that if $N \trianglelefteq G$ and $P \in \mathrm{Syl}_p(N)$, then $G = N_G(P) \cdot N$. [6 marks]

QUESTION 2. (20 MARKS)

(a) Let G be a group and $a, b \in G$. Write $a \sim b$ if and only if there exists a $c \in G$ such that $b = c^{-1}ac$. This relation is called conjugacy. Show that conjugacy is an equivalence class [6 marks]

(b) Let p be prime. Define the following terms as used in group theory.

i. p-subgroup [1 marks]

ii. Sylow p-subgroup [2 marks]

(c) Let G be a finite group, and suppose that $|G| = p^k \cdot m$ such that $(p, m) = 1$ where p is a prime divisor of $|G|$. Then G has an subgroup of order p^k. [11 marks]

QUESTION 3. (20 MARKS)

(a) Suppose G is a group with Z a subgroup of the center $Z(G)$. Prove G is abelian if G/S is cyclic. [5 marks]

(b) Let p be prime and G be a p-group. then $Z(G) > \{1_G\}$. Use this fact and part (a) above to prove that a group of order p^2, where p is prime, is abelian. [3 marks]

(c) Show that D_n, the dihedral group of order $2n$ is solvable for each n. [4 marks]

(d) i. Define a nilpotent group and nilpotence class [4 marks]

ii. Show that every nilpotent group is also solvable. [4 marks]

QUESTION 4. (20 MARKS)

(a) One can show that groups are not isomorphic by showing that a property holds for one group and not the other. Show that $(\mathbb{Q}, +)$ is not isomorphic to $(\mathbb{Z}, +)$.

[6 marks]

(b) Use Sylow's theorems to show that a group of order $48 = 2^3 \cdot 3$ is not simple.

[8 marks]

(c) Use the fundamental theorem of finite abelian groups to classify all abelian groups of order $540 = 2^2 \cdot 3^3 \cdot 5$. [6 marks]

QUESTION 5. (20 MARKS)

(a) Let G_1, G_2, G_3 be groups and let $G = G_1 \times G_2 \times G_3$ be their direct product. Show that G_1 is a subgroup of G and hence $G_1 \trianglelefteq G$ where G_1 is the subgroup associated with the set $\{a, 1_{G_2}, 1_{G_3} : a \in G_1\}$ and $G/G_1 \cong G_2 \times G_3$. [10 marks]

(b) Let $(g, h) \in G \times H$. If g and h have finite orders r and s respectively, Show that the order of (g, h) in $G \times H$ is the least common multiple of r and s. [6 marks]

(c) Find a cyclic subgroup of order 6 in $D_3 \times C_2$. [4 marks]