

## MAS 103: INTRODUCTION TO PROBABILITY THEORY

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### Question 1 [30 Marks]

- a) Distinguish between the subjective and classical approaches to probability [2 Marks]
- b) Suppose two six-sided dice are thrown simultaneously. Construct a table showing all possible events that can occur and use the results to calculate the probability that the sum of the numbers on the two dice will be 5 or less? [2 Marks]
- c) A bag contains 6 red marbles and 4 black marbles. Two marbles are drawn with replacement from the bag. What is the probability that both marbles are black? [2 Marks]
- d) A new MasterCard has been issued to 2000 customers. Of these customers, 1500 hold a Visa card, 500 hold an American Express card and 40 hold a Visa card and an American Express card. Find the probability that a customer chosen at random holds a Visa card, given that the customer holds an American Express card. [2 Marks]
- e) Suppose a coin is tossed four times and  $X$  denotes the number of heads that occur
- Prepare a table showing the probability distribution function of  $X$  [1 Mark]
  - What is the probability of 3 heads or less? [2 Marks]
  - Use the above table to determine the expected value and variance of  $X$  [3 Marks]
- f) Consider an experiment where a coin and a die are tossed.

- i. Are the outcomes of the coin/die throw independent of each other? Justify your answer? [2 Marks]
- ii. Construct a table showing the marginal probabilities for each pair of outcome and the joint probabilities [2 Marks]
- g) Consider a series of 25 independent Bernoulli trials, each of which has probability,  $p=0.4$  of being a 'success'.
- i. Sketch a well-labelled bar chart that describes the distribution of the number of successes. [2 Marks]
- ii. Calculate the probability that the total number of successful trials is exactly 2? [2 Marks]

h) A random variable  $X$  has probability function

$$f(x) = p(X = x) = \frac{1}{2^x} = 1, 2, 3, \dots$$

$$= 0, \text{ elsewhere}$$

Find the moment generating function of  $X$  and hence find the mean and variance of  $X$  [4 Marks]

- i) The fuel consumption per 100 Km of an Isuzu Trooper is known to be normally distributed with mean 18.3 litres and standard deviation 1.2 litres. Suppose you do not want to take a risk of more than 1 in a 100 of running out of fuel, what is the minimum amount of fuel that you should put in your tank for a 100 Km trip? [4 Marks]

### Question 2 [20 Marks]

- a) For the events  $A$  and  $B$ ,  $p(A \cap B') = 0.32$ ,  $p(A' \cap B) = 0.11$  and  $p(A \cup B) = 0.65$
- i) Draw a Venn diagram to illustrate the complete sample space for the events  $A$  and  $B$  [2 Marks]
- ii) Write down the value of  $p(A)$  and  $p(B)$  [2 Marks]
- iii) Determine  $p(A/B')$  [2 Marks]
- iv) Determine whether or not  $A$  and  $B$  are independent [1 Mark]
- b) A discrete random variable  $X$  has the probability function shown in the table below.

$X$	0	1	2
$f(x) = p(X = x)$	$\frac{1}{3}$	$a$	$\frac{2}{3} - a$

- i) Given that  $E(X) = \frac{5}{6}$ , find  $a$  [2 Marks]

ii) Calculate  $\text{Var}(X)$  [2 Marks]

iii) Find the value of  $P(X \leq 1.5)$  [1 Mark]

c) The probability density function of a random variable  $X$  is defined as:-

$$f(x) = 2x \text{ for } 0 < x < 1$$

i) What is its cumulative distribution function? [2 Marks]

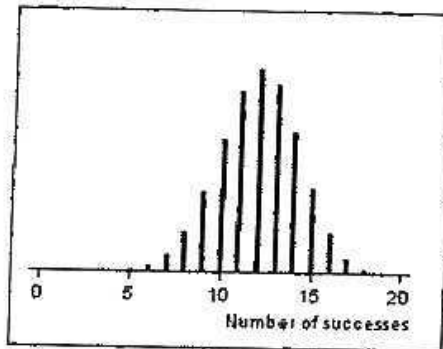
ii) Calculate the expectation of  $X$  and give its meaning [3 Marks]

iii) Calculate the variance of  $X$  [3 Marks]

### Question 3 [20 Marks]

a) Is there a relationship between Bernoulli and binomial distributions? Explain your answer [1 Mark]

b) Consider a series of  $n=20$  independent trials, each of which has probability  $\pi=0.4$  of being a 'success'. Discuss whether the bar chart below correctly describes the distribution of the number of successes? Justify your answer [2 Marks]



c) Consider families with 3 children. Suppose that the proportion of boys in the whole population is 0.512. Assume that the outcome of each birth in the family is independent of previous birth outcomes. Define an appropriate binomially distributed random variable concerning the birth outcomes in a family, and use it to determine the following probabilities:-

i) There will be exactly 3 boys [2 Marks]

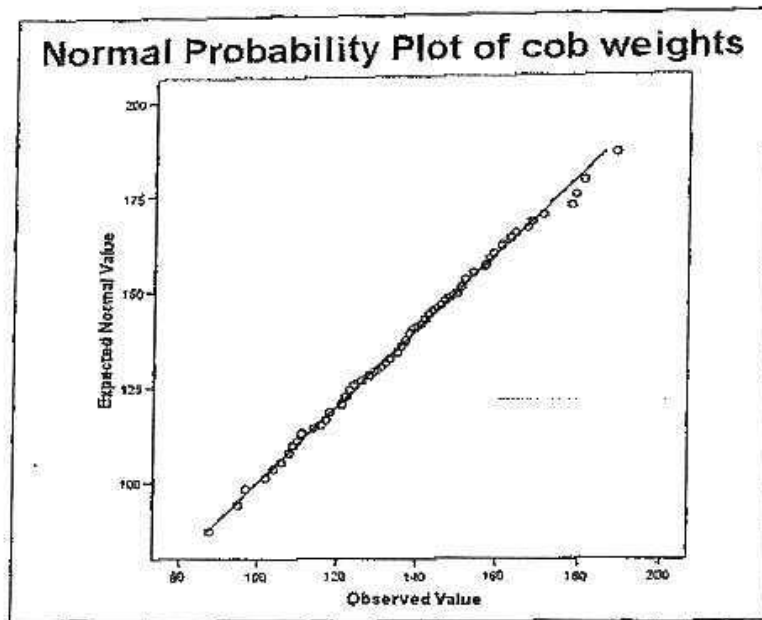
ii) There will be 1 boy and 2 girls [2 Marks]

iii) There will be less than 2 girls [2 Marks]

- d) Find the moment generating function of the binomial distribution with parameters  $n$  and  $p$ . Hence show that the mean  $\mu = np$  and variance  $\sigma^2 = np(1 - p)$  [11 Marks]

#### Question 4 [20 Marks]

- a) Give the probability density function of a random variable that follows a normal distribution [2 Marks]
- b) State whether or not you agree with the statement "All normal distributions have the same shape". Justify your answer with an illustration of the normal curve for different values of the shape parameters [2 Marks]
- c) The pulse rate (say  $X$ ) of healthy individuals is expected to have a normal distribution with mean of 75 beats per minute and a standard deviation of 8. What is the chance that a randomly selected individual will have a pulse rate less than 65? [3 Marks]
- d) The mean yield for one-acre plot is 662 kilos with a standard deviation of 32 kilos. Assuming the normal distribution, how many one-acre plots in a batch of 1000 plots would you expect to have a yield of
- i) Over 700 kilos [3 Marks]
  - ii) Below 650 kilos [3 Marks]
- e) Below is the plot for maize cob weights produced to check the assumption of normality. In this plot, the Y-axis corresponds to values you would expect from an actual normal distribution. The X-axis corresponds to your data. What do you deduce from the graph below? [3 Marks]



f) State four properties of the normal distribution [4 Marks]

**Question 5 [20 Marks]**

a) Two discrete random variables X and Y have the joint probability distribution given in the table below:

		X		
		0	1	2
Y	0	c	0.05	0.10
	1	0.10	0.15	0.15
	2	0.15	0.10	0.10

- i) Find the value of c [2 Marks]
- ii) Find the marginal distributions of X and Y [4 Marks]
- iii) Use your answers in (ii) above to calculate the expectations and variances of X and Y [2 Marks]
- iv) State whether X and Y are independent. Justify your answer [2 Marks]

b) During an epidemic of disease, a doctor sees 110 people who have symptoms commonly associated with a heart condition. Of these, 45 are women, of whom 20 actually have the condition. 15 of the men also have the condition.

- i) Draw a well labelled Venn diagram to represent the scenario [4 Marks]