**UNIVERSITY OF EMBU**

**2017/2018 ACADEMIC YEAR**

**SECOND SEMESTER EXAMINATIONS**

**FOURTH YEAR EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE, BACHELOR OF EDUCATION (SCIENCE) AND BACHELOR OF EDUCATION (ARTS)**

**SMA 420 MATHEMATICAL MODELLING**

**DATE: APRIL 10, 2018** **TIME: 8:30 AM – 10:30 AM**

**INSTRUCTIONS:**

**Answer Question ONE and ANY Other TWO Questions.**

## QUESTION ONE (30 MARKS)

a) Define the following terminologies used in mathematical modelling

i) Mathematical modeling (2 marks)

ii) Carrying capacity (1 mark)

iii) Dynamic modelling (2 marks)

b) Solve for the given initial value problems (5 marks)

$$\frac{dy}{dx} = \frac{1}{y - t^2 y}, \qquad y(0) = 4$$

c) Find the half-life of a radioactive substance with decay equation $y = y_0 e^{-kt}$ and show that the half-life depends only on $k$. (5 marks)

d) At the beginning of the summer, the population of a hive of bald-faced hornets is growing at a rate proportional to the population. From a population of 10 on May 1, the number of hornets grows to 50 in thirty days. If the growth continues to follow the same model, how many days after May 1 will the population reach 100? (5 marks)

e) Find the growth rate and stable stage distribution of the coyote model whose matrix is

$$P = \begin{pmatrix} 0.11 & 0.15 & 0.15 \\ 0.3 & 0 & 0 \\ 0 & 0.6 & 0.6 \end{pmatrix}$$

Will the population grow or decline? (5 marks)

f) For the predator–prey model,

$$P_{t+1} = P_t(1 + 1.6(1 - P_t)) - 0.1P_tQ_t$$

$$Q_{t+1} = 0.3Q_t + 0.6P_tQ_t$$

Compute the equilibria. Which ones are biologically meaningful? (5 marks)

## QUESTION TWO (20 MARKS)

a) Discuss the steps in involved in model formulation (6 marks)

b) According to United Nations data, the world population in 1998 was approximately 5.9 billion and growing at a rate of about 1.33% per year. Assuming an exponential growth model, estimate the world population at the beginning of the year 2023. (4 marks)

c) In 2005 and 2007, the Michigan Department of Natural Resources airlifted 61 moose from Algonquin Park, Ontario to Marquette County in the Upper Peninsula. It was originally hoped that the population P would reach carrying capacity in about 25 years with a growth rate of

$$\frac{dP}{dt} = 0.00003P(2000 - P)$$

i) According to the model, what is the carrying capacity? (2 marks)

ii) With a calculator, generate a slope field for the differential equation. (3 marks)

iii) Solve the differential equation with the initial condition P(0)=61 and show that it conforms to the slope field. (5 marks)

## QUESTION THREE (20 MARKS)

a) Consider the one parameter family model described by the equation:

$$\frac{dy}{dt} = y^3 + \propto y + y$$

i) Locate the bifurcation value and describe the bifurcation that takes place (4 marks)

ii) Draw the bifurcation diagram (5 marks)

b) A bird species lives on two islands, North Island and South Island. On the North Island the population increases through births and deaths by 25% each year whereas on the south Island it increases by through birth and deaths by 10% each year. In addition, each year 400 birds leave the North Island, one-quarter of which go to the South Island.

i) Write down a word equation that defines the population on each Island (2 marks)

ii) Write down, formally, the set of difference equations that model this problem after n years. Define all the variables and explain your terms. (4 marks)

c) Consider the logistic difference equation

$$x_{n+1} = rx_n(1 - x_n)$$

i) Show that the fixed points of the logistic difference equation are $x^* = 0$ and $x^* = (r-1)/r$ (2 marks)

d) Determine the stability of the fixed points as a function of r and explain the biological implications of your answer. (3 marks)

## QUESTION FOUR (20 MARKS)

a) Consider a population described by the differential equation

$$\frac{dy}{dt} = y^3 - 4y + 2$$

i) Find the equilibrium points, their stability and draw the phase portrait (4 marks)

ii) Describe the long-term behavior of the population with the given initial population (2 marks)

b) Solve for the given initial value problems (7 marks)

$\frac{dy}{dt} = \frac{1}{y - t^2 y}$, y(4)=4

c) Consider the difference equation $x_{n+1} = f(x_n)$ $x_0 = X$. Explain how Cobwebbing is used to determine the dynamics of this model for the specified initial condition. (4 marks)

d) A hard-boiled egg at 98°C is put in a pan under running 18°C water to cool. After 5 minutes, the egg's temperature is found to be 38°C. How much longer will it take the egg to reach 20°C? (3 marks)

## QUESTION FIVE (20 MARKS)

In the competition model for two species with populations $N_1$ and $N_2$

$$\frac{dN_1}{dt} = r_1 N_1\left(1 - \frac{N_1}{K_1} - b_{12}\frac{N_2}{K_1}\right),$$

$$\frac{dN_2}{dt} = r_2 N_2\left(1 - b_{21}\frac{N_1}{K_2}\right),$$

where only one species, $N_1$, has limited carrying capacity.

a) Non-dimensionalise the system and determine the steady states. (5 marks)

b) Investigate their stability and sketch the phase plane trajectories. (5 marks)

c) Show that irrespective of the size of the parameters the principle of competitive exclusion holds. (5 marks)

d) Briefly describe under what ecological circumstances the species $N_2$ becomes extinct. (5 marks)

**--END---**