UNIVERSITY OF EMBU

2016/2017 ACADEMIC YEAR

SECOND SEMESTER EXAMINATION

FOUTH YEAR EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE

SMA 403: MEASURE THEORY

DATE: APRIL 12, 2017 **TIME: 11:00AM-1:00PM**

INSTRUCTIONS:

Answer Question ONE and ANY Other TWO Questions.

QUESTION ONE (30 MARKS)

a) Explain what is meant by measure as used in measure theory. (2 marks)

b) Consider $(X, P(X))$ where (P(X) is power set of set X.

Define $\mu: P(X) \rightarrow \mathbb{R}$ by

$$\mu(E) = \begin{cases} 1 & if\ b \in E \\ 0 & iF\ b \notin E \end{cases}$$

Determine whether μ is a measure. (4 marks)

c) Let F be a measureable real valued function defined on X and $\in \mathbb{R}$. Show that cf is measurable. (4 marks)

d) Let μ be a measure defined on δalgebra of non empty set x, prove that if $E, F \in \mathcal{A}$ and E$\subset f$ then $\mu(E) \leq \mu(F)$ (3 marks)

e) Show that a singleton set is measureable and its measure is zero. (2 marks)

f) Prove that F is integrable if and only if |F| is integrable. (3 marks)

g) Let $f, g \in m^*(X,)$ with $f \leq g$, then prove that $\int f d\mu \leq \int g d\mu$ (3 marks)

h) If f is lebesque measurable and $M^*(F\Delta G) = 0$, then prove that G is lebesque measureable.

(4 marks)

i) Let $f(x) = \begin{cases} 25 & IF\ 1 \leq x < 4 \\ 8 & IF\ 4 \leq x \leq 15 \\ 0 & elsewhere \end{cases}$ Evaluate the integral by using Lebesque measure.

(5 marks)

QUESTION TWO (20 MARKS)

a) Define the lebesque outer measure m^* and show that it is translation invariant. (7 marks)

b) Explain what is meant by "a subset of E of $\mathbb{R}$ is lebesque measurable" and prove that every interval in $\mathbb{R}$ is lebesque measurable. (7 marks)

c) If $f, g \in M^*(X, \mathcal{A})$ and $f + g \in M^*(X, \mathcal{A})$ then prove that

$\int (f + g) d\mu = \int f d\mu + \int g d\mu$ (6 marks)

QUESTION THREE (20 MARKS)

a) Let $(X, \mathcal{A})$ be a measurable space and $f . X \to \mathbb{R}$ be $\mathcal{A}$ − measurable function. Prove that the following statements are equivalent for $\propto \in \mathbb{R}$

(i) $A_{\propto} = \{x \in X : F(x) > \propto\} \in \mathcal{A}$

(ii) $B_{\propto} = \{x \in X : F(x) \leq \propto\} \in \mathcal{A}$

(iii) $C_{\propto} = (x \in X : F(x) \geq \propto\} \in \mathcal{A}$

(iv) $D_{\propto} = \{x \in X : F(x) < \propto\} \in \mathcal{A}$ (10 marks)

b) Suppose that f is a measurable real valued function defined on X, prove that F^2 is also measurable. (4 marks)

c) Suppose that $f \in M^*(X, A) then\ f(x) = 0$ μ almost everywhere on $\mathcal{A}$ if and only if $\int f d\mu = 0$

(6 marks)

QUESTION FOUR (20 MARKS)

a) State and prove the monotone convergence theorem. (9 marks)

b) (i) State without proof Fatou's Lemma (2 marks)

(ii) Give an example that shows that we can have strict inequality in Fatou's lemma

(4 marks)

c) Prove that convergence in Lp implies convergence in measure (5 marks)

QUESTION FIVE (20 MARKS)

a) Show that the cantor set is uncountable set with Lebesque measure zero. (7 marks)

b) Let $\{f_n\}$ be a sequence of integrable functions which converges almost everywhere to a real values measureable function f. If there exists an integrable function of g such that $|fn| \leq g$ $\forall n$ prove that f is integrable and

$\int(\lim_{n\to\infty} f_n)d\mu = \int f d\mu = \lim_{n\to\infty} \int f_n d\mu$ (9 marks)

c) Let f, g be integrable function on X. Prove that $f + g$ is also integrable. (4 marks)

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