UNIVERSITY OF EMBU

2016/2017 ACADEMIC YEAR

SECOND SEMESTER EXAMINATION

THIRD YEAR EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE AND BACHELOR OF EDUCATION (SCIENCE AND ARTS)

SMA 303: ALGEBRA 1

DATE: APRIL 13, 2017 **TIME: 8:30-10:30 AM**

INSTRUCTIONS:

Answer Question ONE and ANY Other TWO Questions.

QUESTION ONE (30 MARKS)

a) Define a group G with binary operation $*$. (3 marks)

b) Let x and y be elements of a group G . Show that $(xy)^{-1} = y^{-1}x^{-1}$. (3 marks)

c) Let x be an element of a group G. Show that the set of elements of G of the form x^n for some integer n is a subgroup of G. (4 marks)

d) Prove that the kernel of a group homomorphism is a normal subgroup. (4 marks)

e) Assume that the equation $xyz = 1$ holds in a group G. Does it imply that $yzx = 1$? that $yxz = 1$? justify your answer. (4 marks)

f) Let G be a group and let $a \in G$. Show that if $a^n = e$ for $n \neq 0$ then $o(a)$ divides n. (4 marks)

g) Show that a group cannot be written as the set-theoretical union of two proper subgroups. (4 marks)

h) Show that every division ring is an integral domain. (4 marks)

QUESTION TWO (20 MARKS)

a) Let S be the set of all real numbers of the form $a + b\sqrt{2}$ where $a, b \in \mathbb{Q}$ are not simultaneously zero. Show that S becomes a group under the usual multiplication of real numbers. (8 marks)

b) Show that a finite intersection of subgroups of a group G is a subgroup of G. (8 marks)

c) Let $f : R \to S$ be a homomorphism of a ring R to a ring S. Show that if R is commutative, then $f(R)$ is commutative (4 marks)

QUESTION THREE (20 MARKS)

a) State without proof the homomorphism theorem. (2 marks)

b) Let the maps $\phi : G \to H$ and $\varphi : H \to K$ be homorphisms. Prove that;

i. The composition $\varphi o \phi : G \to K$ is a homomorphism. (3 marks)

ii. If $\phi : G \to H$ and $\varphi : H \to K$ are isomorphisms, then the composition $\varphi o \phi : G \to K$ is an isomorphism. (6 marks)

iii. If $\phi : G \to H$ is an isomorphism, then $\phi^{-1} : H \to G$ is an isomorphism. (3 marks)

c) Show that for any nonempty subset S of a group G, the normalizer of S is a subgroup of G. (6 marks)

QUESTION FOUR (20 MARKS)

a) State without proof Lagrange's theorem for groups. (2 marks)

b) Prove that every group of prime order is cyclic. (4 marks)

c) Let H be a normal subgroup of G. Show that the following statements are equivalent

i. $H \lhd G$.

ii. $xHx^{-1} = H$.

iii. $xH = Hx$.

iv. $xHyH = xyH$. (8 marks)

d) Determine whether the following permutations are even or odd.

i. (123)(134) (3 marks)

ii. (1456)(215) (3 marks)

QUESTION FIVE (20 MARKS)

a) Prove that the ring $\mathbb{Z}$ of integers is a principal ideal domain. (5 marks)

b) `Show that although is $\mathbb{Z}_2$ an integral domain, the matrix ring $M_2(\mathbb{Z}_2)$ has divisors of zero. (3 marks)

c) Let G be a group and $g \in G$, define a map $\varphi_g : G \to G$ by $\varphi(x) = gxg^{-1}$. Show that φ_g is an automorphism of G determined by g. (6 marks)

d) Let H be a normal subgroup of G, show that the mapping $\varphi : G \to G/H$ defined by $\varphi(x) = xH$ is surjective homomorphism and $Ker\varphi = H$. (6 marks)

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