UNIVERSITY OF EMBU

2016/2017 ACADEMIC YEAR

FIRST SEMESTER EXAMINATION

THIRD YEAR EXAMINATION FOR THE DEGREE OF BACHELORS OF SCIENCE AND BACHELOR OF EDUCATION (SCIENCE & ARTS)

SMA 301: REAL ANALYSIS I

DATE: DECEMBER 7, 2016 **TIME: 2:00-4:00PM**

INSTRUCTIONS:

Answer Question ONE and ANY other TWO Questions

QUESTION ONE (30 MARKS)

a) Let S be a subset of a metric space M. Define the following terms

i. A limiting point of S (1 mark)

ii. A neighborhood of a point x in M (1 mark)

iii. Closure of S (1 mark)

iv. Interior of S (1 mark)

v. A dense subset S of M (1 mark)

b) In each of the following sets, determine whether the set is closed or open

i. $S = \{(-1)^n : n \in N\}$ (2 marks)

ii. $S = \left\{(-1)^n \ \frac{n+1}{n} : n \in N\right\} \cup \{-1,1\}$ (2 marks)

c) i Show that set $(2, 3) = G$ is open in $\mathbb{R}$ (2 marks)

ii Prove that arbitrary intersection of closed sets is closed (3 marks)

iii. Show that if N_1 and N_2 are neighborhoods of p, then their intersections is also a neighborhood of p. (3 marks)

d) i) When is a subset A said to be compact on a metric space (X, d)? (1 mark)

ii) Show that any open interval (a, b) is not compact (2 marks)

e) i) State without proof the Holder's inequality as applied in metric spaces (2 marks)

ii) Define a function $d: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by $d(x, y) = \exp\{|x - y|\}$. Determine whether d is a metric. (4 marks)

f) Show that$f(x) = \frac{1}{x}$ is continous on the interval $0 < x < 1$ but not uniformly continuous on the same interval. (4 marks)

QUESTION TWO (20 MARKS)

a) Define a complete metric space. Hence illustrate whether a set of rational numbers is complete or not. (3 marks)

b) Prove that Intersection of finitely many open sets in a metric space is open. (7 marks)

c) Prove that the complement of an open set is closed and the complement of a closed set is open (10 marks)

QUESTION THREE (20 MARKS)

a) i) Let $g: X \rightarrow Y and\ f: Y \rightarrow Z$ be continuous functions on the metric spaces X, Y and Z. Using the f^{-1} definition, prove that then their composite function $(f\ 0\ g)$ is continuous. (3 marks)

b) ii) Show that $\lim_{x \rightarrow x_0 x^2 =}\ x_0^2$ (4 marks)

c) Prove that in any metric space (X, d),

i) An open sphere is a neighborhood of each and every point in it (6 marks)

ii) An empty set is an open subset of X (2 marks)

d) $\forall\, x, y \in R^n$, define a function $d: R^n \times R^n \to \mathrm{R}$ by $(x, y) = \left(\left(\sum_{i=1}^{n}(x_i - y_i)^2\right)\right)^{\frac{1}{2}}$. Show that the metric space (R^n, d) is complete. (5 marks)

QUESTION FOUR (20 MARKS)

a) State without proofs;

(i) Cauchy-Schwartz inequality (2 marks)

(ii) Minkowski's inequality on R^n (2 marks)

b) $\forall\, x \in \iota^{\infty}: x = (x_k)_k{}^{\infty} = (x_1, x_2, \ldots)$. $\forall\, x, y \in \iota^2$, define a function $d: \iota^2 \times \iota^2 \to \mathrm{R}$ by

$$d(x, y) = \left(\sum_{k=1}^{\infty} |x_k - y_k|^2\right)^{\frac{1}{2}}$$

Show that (ι^2, d) is a metric space. (10 marks)

c) Prove that every closed subset of a compact metric space is compact (6 marks)

QUESTION FIVE (20 MARKS)

a) Determine whether the function $f(x) = \ln x, \quad 0 < x < 1$ is uniformly continuous or not.
(Hint: Take $x'_n = e^{-n}$, $x'_n = e^{-n-1}$) (4 marks)

b) Show that $f(x) = x^2$ is uniformly continuous on the interval $[0, 5]$ but not uniformly continuous on the interval $(0, \infty)$ (3 marks)

c) Using appropriate examples, explain the three different types of discontinuities of a function (6 marks)

d) Show that a function which is uniformly continuous on metric space is continuous on that metric space (3 marks)

e) State and illustrate the proof of Cantor's theorem. (4 marks)

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