JOMO KENYATTA UNIVERSITY
OF
AGRICULTURE AND TECHNOLOGY
UNIVERSITY EXAMINATIONS 2018/2019
EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE

STA 2300: THEORY OF ESTIMATION

DATE: December 10, 2018 **TIME: 2 HOURS**

INSTRUCTIONS TO CANDIDATES

1. *Answer question ONE (entire section A) and any other two questions in section B.*
2. *Show all your workings, since all steps leading to presented solutions will attract marks.*
3. *No writing is allowed on the question paper which has four printed pages.*

SECTION A (30 MARKS)

1. (a) Let $f(x,\theta)$ for $\theta \in \Omega$ be some *pdf*(*pmf*) and suppose that $\hat{\theta}_1$ and $\hat{\theta}_2$ are possible independent estimators of θ, such that $E[\hat{\theta}_1] = E[\hat{\theta}_2] = \theta$, $var(\hat{\theta}_1) = \sigma_1^2$ and $var(\hat{\theta}_1) = \sigma_2^2$. A new estimator $\hat{\theta}_3$ is defined as $\hat{\theta}_3 = a\hat{\theta}_1 + (1-a)\hat{\theta}_2$.

   i. Examine $\hat{\theta}_3$ for unbiasedness.

   ii. How should the constant a be chosen in order to minimise the variance of $\hat{\theta}_3$. **[4 marks]**

(b) The reading on a voltage meter connected to a test circuit is uniformly distributed over the interval $(\theta, \theta + 1)$, where θ is the true but unknown voltage of the circuit. Suppose that $Y_1, Y_2, \ldots, Y_n$ denotes a random sample of such readings.

i. Show that $\bar{Y}$ is a biased estimator of θ and compute the bias.

ii. Find a function of $\bar{Y}$ that is an unbiased estimator of θ.

iii. Find the Mean Squared Error of $\bar{Y}$, when $\bar{Y}$ is used as an estimator of θ.

[4 marks]

(c) Suppose that we are to obtain a single observation Y from an exponential distribution with mean θ. Use Y to form a confidence interval for θ with confidence coefficient of 0.90. [3 marks]

(d) Suppose that $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ are independent random samples from populations that are normally distributed with means μ_1, μ_2 and variances σ_1^2, σ_2^2 respectively. Show that;

i. $\bar{X} - \bar{Y}$ is a consistent estimator of $\mu_1 - \mu_2$.

ii. $\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2+(Y_i-\bar{Y})^2}{2n-2}$ is a consistent estimator of σ^2. [4 marks]

(e) Let $Y_1, Y_2, \ldots, Y_n$ denote independent and identically distributed random variables from a power family distribution with parameters α and θ where $\alpha > 0$ and $\theta > 0$ and $f(y/\alpha, \theta) = \frac{\alpha y^{\alpha-1}}{\theta^\alpha}$ for situations where $o \le y \le \theta$. Suppose θ is known;

i. Find the moments estimator of α.

ii. Identify a complete and sufficient statistic for α. Is the identified statistic a Uniformly Minimum Variance Unbiased Estimator (UMVUE) of α? Explain your answer. [4 marks]

(f) Suppose that $Y_1, Y_2, \ldots, Y_n$ denotes a random sample from the Weibul density function given by $f(y, \theta) = (\frac{2y}{\theta})e^{\frac{-y^2}{\theta}}$ when $y > 0$. Find the Minimum Variance Bound Unbiased Estimator (MVBUE) for θ. [4 marks]

(g) Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a density given by $f(y, \theta) = (\frac{1}{\theta})ry^{r-1}e^{\frac{-y^r}{\theta}}$ for $\theta > 0$ and $y > 0$, where r is a known positive constant.

i. Find a sufficient statistic for θ.

ii. Find the Maximum Likelihood Estimator (MLE) of θ.

iii. Is the MLE a Minimum Variance unbiased Estimator (MVUE) for θ?.

[4 marks]

(h) Let X_1 and X_2 be a random sample of size 2 from a binomial distribution where the probability of success, p, is not known. Assume that the prior of p is given $h(p) = 1$ for $0 < p < 1$. Find;

i. the posterior distribution of p.

ii. the Bayes Estimator of p. [3 marks]

SECTION B (20 MARKS EACH)

2. Let $Y_1, Y_2, \dots, Y_n$ denote a random sample from the probability density function $f(y, \theta) = \theta y^{\theta-1}$ for $0 < y < 1$ and $\theta > 0$.

   (a) Show that this density function is a one-parameter exponential family and that $\sum_{i=1}^{n} -ln(Y_i)$ is complete and sufficient for θ.

   (b) If $W_i = -ln(Y_i)$, show that $W_i \sim exp(\frac{1}{\theta})$.

   (c) Show that $2\theta \sum_{i=1}^{n} W_i \sim \chi^2_{(2n)}$ and that $E(\frac{1}{2\theta \sum_{i=1}^{n} W_i}) = \frac{1}{2(n-1)}$.

   (d) Find the Minimum Variance unbiased Estimator (MVUE) for θ.

3. (a) Let $Y_1, Y_2, \dots, Y_n$ denote a random sample from a population with mean μ and variance σ^2. Consider the following three estimators for μ.
   $\hat{\mu}_1 = \frac{1}{2}(Y_1 + Y_2)$, $\hat{\mu}_2 = \frac{1}{4}Y_1 + \frac{Y_2 + Y_3 + \dots + Y_{n-1}}{2(n-2)} + \frac{1}{4}Y_n$ and $\hat{\mu}_3 = \bar{Y}$.

   i. Show that each of these three estimators is unbiased for μ.

   ii. Find the Efficiency of $\hat{\mu}_3$ relative to $\hat{\mu}_2$ and $\hat{\mu}_1$ respectively.

   (b) Suppose that $Y_1, Y_2, \dots, Y_n$ constitute a random sample from a population with probability density function $f(y) = (\frac{1}{\theta+1})e^{\frac{-y}{\theta+1}}$ when $y > 0$ and $\theta > -1$.

   i. Is θ estimable in this case?

   ii. Assuming that θ is estimable in part 3(b)i, find the Frechet-Crammar-Rao Lower Bound for the unbiased estimators of θ and hence find the most efficient estimator of θ.

4. A random variable X has a density function given by $f(x/\theta) = \frac{1}{\theta}$ for $0 < x < \theta$, where θ is not known.

(a) Find the Maximum Likelihood Estimator (MLE) of θ.

(b) Suppose that the prior distribution of the said unknown θ is $h(\theta) = \theta e^{-\theta}$ for $\theta > 0$. By considering a quadratic loss function $L(d(x), \theta) = c(\theta)[d(x) - \theta]^2$ with $c(\theta) > 0$ and a single observation. Find the decision function which minimises the Bayes Risk.

(c) From the usual definitions and meanings, distinguish a MINIMAX estimator from a Bayes estimator, and use your results in part 4b to elaborate this difference if such exists.

THE END